Solving a 3×3 system of equations by Gaussian Elimination - Excel Spreadsheet

This document provides a guide to the Excel spreadsheet[1] for solving a matrix-vector equation with three unknowns by Gaussian elimination[2]. The spreadsheet solves a system of the form

$$A\underline{x} = \underline{b}\,,$$

where A is a 3×3 matrix and $\underline{x}$ and $\underline{b}$ are 3×1 vectors. A and $\underline{b}$ are stated and the solution $\underline{x}$ is found via the spreadsheet.

The spreadsheet image below shows the solution of the following system

$$\begin{pmatrix} 2 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 2 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix}$$

Which is the matrix-vector form of the following system of equations:

$$2x_1 + x_2 + 2x_3 = 3$$

$$x_1 + x_3 = 1$$

$$2x_1 + 2x_2 - x_3 = -2\,.$$

These are also known as simultaneous equations. There are several methods for solving systems of two simultaneous equations in two unknowns and these can sometimes be extended to solve a 3x3 system. For example, for this system, we could use elimination[3]; we could eliminate x_2 by subtracting twice the first equation from the third. We would then have two equations in x_1 and x_3 only and could then progress towards the solution. Alternativel, we could use substitution[4]; write the second equation as $x_1 = 1 - x_3$ and substitute for x_1 in the first and third equation and again we have two simultaneous equations in two unknowns and could then work toward the solution. However, although these methods could yield the correct solution, Gaussian elimination must be a systematic, rather than ad-hoc method. Gaussian elimination is an algorithm that can be implemented in computer.

The matrix A and the vector $\underline{b}$ are placed in the yellow cells, the working out and solution is below the thick black line. The solution is in the blue cells. The method proceeds from the first row to the penultimate row, On each row the diagonal element acts as the pivot and by subtracting a proportion of that row from the rows below, each value below the pivot is eliminated, that is to become zero, Eventually the matrix is upper triangular.

[1] Gaussian Elimination (Spreadsheet)
[2] Gaussian Elimination
[3] Solving Simultaneous Equations by Elimination
[4] Solving Simultaneous Equations by Substitution

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Finding the solution of a 3x3 matrix-vector equation by Gaussian Elimination

Guide

Matrix (A) and vector (b) -	2	1	2				3
place in yellow area	1	0	1	X	x	=	1
	2	2	-1				-2

		2	1	2	3	matrix and vector
		1	0	1	1	
		2	2	-1	-2	
		2	1	2	3	eliminate (2,1), (3,1)
row 2 -	0.5 x row 1	0	-0.5	0	-0.5	
row 3 -	1 x row 1	0	1	-3	-5	
		2	1	2	3	eliminate (3,2)
		0	-0.5	0	-0.5	
row 3 -	-2 x row 2	0	0	-3	-6	
	SOLUTION x=				-1	back-substitution
					1	
					2	

For the first column, below the pivot '2' on the diagonal, the '1' can be eliminated by subtracting 0.5 of row 1 from row 2:

$$[1 \quad 0 \quad 1 \quad 1]-0.5\,[2 \quad 1 \quad 2 \quad 3]=[0 \quad -0.5 \quad 0 \quad -0.5]\,.$$

Similarly subtracting row 1 from row 3 eliminates the '2' on the first column of row 3:

$$[2 \quad 2 \quad -1 \quad -2]-1[2 \quad 1 \quad 2 \quad 3]=[0 \quad 1 \quad -3 \quad -5]\,.$$

Rows 2 and 3 of the original matrix and vector are replaced by the new rows that have been attained by row operations.

Finally subtracting -2x row 2 from row 3 eliminates the '1' on the second column and third row of the matrix:

$$[0 \quad 1 \quad -3 \quad -5]--2[0 \quad -0.5 \quad 0 \quad -0.5]=[0 \quad 0 \quad -3 \quad -6]\,,$$

The matrix is now upper-triangular and *back substitution* is applied in order to complete the solution.

$$\begin{pmatrix} 2 & 1 & 2 \\ 0 & -0.5 & 0 \\ 0 & 0 & -3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3 \\ -0.5 \\ -6 \end{pmatrix}$$

This involves starting with the final row: $-3x_3 = -6$ hence $x_3 = 2$. Then going up one row: $-0.5\, x_2 = -0.5$ and hence $x_2 = 1$. The first equation is $2x_1 + x_2 + 2x_3 = 3$, susbtituting the value for x_2 and x_3 gives $x_1 = -1$.

Notes/Exercises

Pivoting

If we try to solve the following problem

$$\begin{pmatrix} 0 & 2 & 2 \\ 1 & 0 & 1 \\ 1 & 2 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \\ 3 \end{pmatrix}$$

the method runs into problems, eventhough the matrix-vector equation has a unique solution (the determinant of the matrix is 4). The reason for the problem is that as the method of Gaussian elimination moves through the columns, the row operations require division by the diagonal element (the '*pivot*'). In the case of this example, the first diagonal element is 0 and hence the method fails.

Hence the implementation of the method of Gaussian elimination, the method is accompanied by a method called *partial pivoting* (or *full pivoting*). Partial pivotting involves inspecting the elements below the diagonal that is being worked on and finding the largest value and exchanging that row with the row coincident with the relevant diagonal element. In the example above, the initial application of partial pivoting would suggest swapping rows 1 and 2 to give.

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 2 \\ 1 & 2 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \\ 3 \end{pmatrix},$$

which returns the solution. (Note that full pivoting involves also exchanging columns to achieve the same (this also has the effect on the order of the $\underline{x}$-vector)).

Singular matrix – no solution

In order to investigate the response to a singular matrix. The matrix in the following system is singular. The spreadsheet is unable to solve this, the bottom row of the eliminated matrix is all zeros and the bottom row of the vector is non-zero; this is inconsistent and a message 'no solution' is posted.

$$\begin{pmatrix} 0 & 2 & 2 \\ 1 & 0 & 1 \\ 1 & 2 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \\ 3 \end{pmatrix}$$

Singular matrix – many solutions

In order to investigate the response to a singular matrix. The matrix in the following system is singular. The spreadsheet is unable to solve this, the bottom row of the eliminated matrix is all zeros and the bottom row of the vector is zero; the bottom equation gives us no further information on top of the first two equations and the message 'many solutions' is posted.

$$\begin{pmatrix} 0 & 2 & 2 \\ 1 & 0 & 1 \\ 1 & 2 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \\ 6 \end{pmatrix}$$